

Thermal conductivity due to phonons in the core of superfluid neutron stars

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We compute the contribution of phonons to the thermal conductivity in the core of superfluid neutron stars. We use effective field theory techniques to extract the phonon scattering rates, written as a function of the equation of state of the system. We also calculate the phonon dispersion law beyond linear order, which depends on the gap of superfluid neutron matter. With all these ingredients, we solve the Boltzmann equation numerically using a variational approach. We find that the thermal conductivity κ is dominated by combined small and large angle binary collisions. As in the color-flavor-locked superfluid, we find that our result can be well approximated by $\kappa \propto 1/\Delta^6$ at low T , where Δ is the neutron gap, the constant of proportionality depending on the density. We further comment on the possible relevance of electron and superfluid phonon collisions in obtaining the total contribution to the thermal conductivity in the core of superfluid neutron stars.

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I. INTRODUCTION

In this paper we compute the superfluid phonon contribution to the thermal conductivity (κ) in the core of superfluid neutron stars. This is a follow up of other publications where the superfluid phonon contribution to the shear viscosity [1] or bulk viscosity coefficients [2] for this system have been computed, with the aim of assessing its effect on the dynamics of the hydrodynamical modes of the star [3].

The cooling of a neutron star depends on the rate of neutrino emission, the photon emission rate, the specific heat and also in young stars on the thermal conductivity [4, 5]. As all these quantities depend on the microscopic processes occurring inside the star, it has been recognized that the measurement of the cooling rate of a neutron star could give some hints on its composition. That is why computing the thermal conductivity in a compact star is so relevant. Further, the thermal conductivity is also needed to get the relaxation of any heat flux produced by any dissipative process in the star.

So far the thermal conductivity in the core of neutrons stars has been computed assuming both normal and superfluid neutron matter phases [6–8]. In the last case, it has been claimed that the thermal conductivity is dominated by electron collisions (and muons, for sufficiently large densities inside the core of the star). In this paper we compute the contribution to the thermal conductivity of the superfluid phonon, which is a collective mode which appears due to the onset of superfluidity. We leave for a future project a careful study of the interactions of the superfluid phonons with the electrons, and their impact on the transport properties of the core of neutron stars.

We use effective field theory techniques to assess the relevant superfluid phonon scattering amplitudes. It is known that at leading order (LO) in an energy and momentum expansion the self-phonon interactions can be determined with the knowledge of the equation of state (EoS) of the superfluid [9]. We use this fact to determine the collision term that enters the Boltzmann equation, which we solve in order to determine the value of the thermal conductivity κ . Technically, the problem is totally analogous to the computation of the superfluid phonon contribution to the thermal conductivity in color-flavor locked quark matter [10], or for the cold Fermi atoms in the unitarity limit [11], the main difference arises from the fact that, since the systems are different, the value of the phonon couplings and dispersion laws differ.

In this work, as in our previous studies, we consider a simplified model of neutron star made up by neutrons, protons and electrons, using a causal parametrization of the Akmal, Pandharipande and Ravenhall (APR for short)

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EoS [12, 13] to describe β -stable nuclear matter inside the star, and get from it all the phonon self-couplings. For the superfluid gap, we use two different parametrizations in order to consider the cases of strong and weak superfluidity [14]. We also note, as previously found out for other superfluid systems [10, 15], that if we take into account the phonon dispersion law at LO the thermal conductivity strictly vanishes. Knowledge of the phonon dispersion law beyond LO is then needed. Thus, we first compute such a correction to the linear dispersion law by considering s-wave neutron pairing, and assume that the phonon dispersion law remains the same in the p-wave phase that is supposed to prevail in most part of the superfluid core.

At this stage one should mention that superfluid phonons may be overdamped or even disappear for some temperatures T [16, 17]. Leggett determined the overdamping of these collective modes in superfluid fermionic systems by analyzing the associated Landau parameters within the Fermi liquid theory. However, for the present case of the APR EoS, we cannot explore this possibility since we cannot extract the corresponding Landau parameters as the nucleon-nucleon effective interaction is not available for the given EoS model. Thus, we will not take into account the possible disappearance of superfluid phonons leaving for a future project an alternative independent study on the overdamping of superfluid phonons in the interior of neutron stars.

This paper is structured as follows. In Sec. II we discuss the relevant ingredients of the phonon dynamics for our computation. In Sec. II A we obtain the superfluid phonon dispersion law beyond linear order, and in Sec. II B we show the relevant phonon collisions which contribute to the thermal conductivity. In Sec. III A we explain the variational approach we use to solve the Boltzmann equation, which is the same as in Ref.[10]. In Sec. III B we present some arguments to derive the scaling behavior of the thermal conductivity with the temperature and gap, respectively, and in Sec. III C we show our numerical results. In Sec. IV we present a comparison between the electron-muon and phonon contributions to the thermal conductivity and discuss that a careful study of the electron-phonon interactions is still needed to fully determine the value of κ . We use natural units ($\hbar = c = k_B = 1$) throughout the manuscript, although we present our results in c.g.s. units.

II. SUPERFLUID PHONONS IN THE CORE OF NEUTRON STARS

The superfluid phonon is a Goldstone mode which appears due to the spontaneous symmetry breaking of the $U(1)$ symmetry induced by the appearance of a difermion condensate. The dynamics of the phonon can be completely determined from the microscopic physics, if one integrates out the heavy degrees of freedom of the system. The basic phonon interaction rates can be derived using effective field theory techniques. The effective phonon Lagrangian is presented as an expansion in derivatives (or momenta and frequencies) of the phonon field, the terms of this expansion being restricted by symmetry considerations. The coefficients of the Lagrangian can be computed from the microscopic theory, through a standard matching procedure, and thus they depend on the short range physics of the system under consideration. The effective field theory assumes that the phonon momentum k and frequency k_0 are such that $kv_F, k_0 \ll \Delta$, where v_F is the Fermi velocity. The expansion parameter of the theory is not a coupling constant, but rather the values of kv_F/Δ and k_0/Δ .

At LO in the derivative expansion, all the phonon self-couplings can be parametrized in terms of the speed of sound, the density of the superfluid, and derivatives of the speed of sound with respect to the density at $T = 0$ [18]. Thus, the phonon physics at leading order is completely determined by the EoS of the superfluid. In this work, as in our previous studies, we consider a model of neutron star made up by neutrons, protons and electrons, using a causal parametrization of the APR EoS [12, 13]. This EoS is widely used as benchmark for β -stable neutron star matter and, from this EoS, we obtain all the phonon self-couplings. We refer the reader to Ref. [1] for more explicit details of the phonon Lagrangian and EoS used in this work.

For the computation of the superfluid phonon contribution to the thermal conductivity, the phonon dispersion law at next to leading order (NLO) is however required. This is so, as the thermal conductivity strictly vanishes if one uses a linear dispersion law. In this Section we compute the first correction beyond linear order to the phonon dispersion law, and later on we show the phonon collisions that we consider in the computation of the thermal conductivity.

A. Superfluid phonon dispersion law beyond linear order

In this Subsection we will compute the phonon dispersion law, assuming s-wave pairing of the neutrons. To study the main properties of the superfluid system, it is enough to consider a Lagrangian keeping only the interaction that is responsible for the formation of the gap, after integrating out all the modes away from Fermi surface [19, 20]. The relevant Lagrangian then reads

$$\mathcal{L}_N = \psi_n^\dagger (i\partial_t - \frac{\nabla^2}{2m^*} + \mu) \psi_n - \frac{G}{2} (\psi_n^\dagger(x) \psi_n(x)) (\psi_n^\dagger(x) \psi_n(x)) + \dots \quad (1)$$

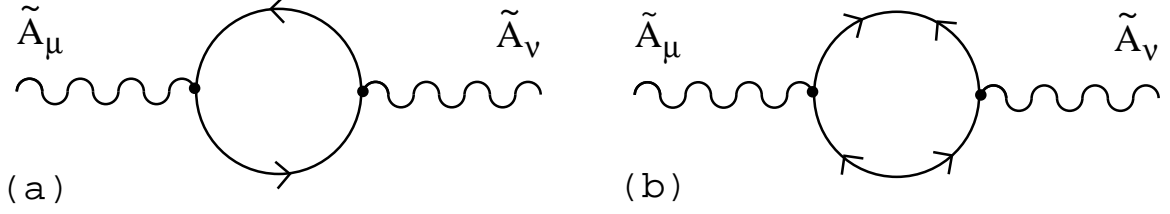


FIG. 1: One-loop Feynman diagrams needed for the computation of the phonon dispersion law. In a) and b) the diagrams correspond to the diagonal/off-diagonal terms in the Nambu-Gorkov basis, respectively.

where ψ_n is the neutron wave-function and μ is the chemical potential associated to the neutrons. Here m^* and G should be matched with the complete microscopic theory to correctly reproduce the value of the Fermi momentum and gap, respectively. One can write $\psi_n = \psi_{n,0} e^{i\varphi}$, where the phase of the neutron wave-function is the Goldstone field. The kinetic term associated to $\psi_{n,0}$ is the same as that for the whole neutron wave-function, where one only has to substitute the standard derivatives by covariant derivatives

$$\psi_n^\dagger (i\partial_t - \frac{\nabla^2}{2m^*} + \mu) \psi_n = \psi_{n,0}^\dagger (i(\partial_t + i\tilde{A}_0) - \frac{(\nabla + i\tilde{\mathbf{A}})^2}{2m^*} + \mu) \psi_{n,0} , \quad (2)$$

where $\tilde{A}_\mu = (\partial_t \varphi, \nabla \varphi)$ is written in terms of the Goldstone field φ . Integrating out the field $\psi_{n,0}$ one is left with an effective field theory for the superfluid phonon, from which its dispersion law can be obtained.

For this computation we use the Nambu-Gorkov formalism (for a review see, e.g., Ref. [21]). In the Nambu-Gorkov basis

$$\chi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ \psi_c \end{pmatrix} , \quad (3)$$

where $\psi_c = C\psi^*$ and $C = i\sigma_2$ where σ_2 is a Pauli matrix, the fermion propagator then becomes

$$S(p_0, \mathbf{p}) = \frac{1}{p_0^2 - \xi_p^2 - \Delta^2} \begin{pmatrix} p_0 + \xi_p & \Delta \\ \Delta^* & p_0 - \xi_p \end{pmatrix} , \quad (4)$$

where $\xi_p = \frac{p^2}{2m^*} - \mu$, and Δ is the gap parameter.

The phonon dispersion law is then obtained by studying the solutions to the equation $k^\mu k^\nu \Pi_{\mu\nu}(k_0, \mathbf{k}) = 0$, where $\Pi_{\mu\nu}$ is obtained after computing (see the one-loop Feynman diagrams displayed in Fig. 1)

$$\Pi_{\mu\nu}(k_0, \mathbf{k}) = -i \int \frac{d^4 p}{(2\pi)^4} \text{Tr}_{NG} [\Gamma_\mu S(p_0, \mathbf{p}) \Gamma_\nu S(p_0, -k_0, \mathbf{p} - \mathbf{k})] , \quad (5)$$

where Tr_{NG} is a trace performed on the Nambu-Gorkov indices, and $\Gamma_\mu = (\Gamma_0, \Gamma_i)$ is the vertex factor for the coupling of the \tilde{A}_μ field in the Nambu-Gorkov basis, and can be expressed as

$$\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \Gamma_i = \frac{1}{2m^*} \begin{pmatrix} p_i & 0 \\ 0 & p_i \end{pmatrix} . \quad (6)$$

The computation of the one-loop diagram can be carried out analytically if one performs an expansion in the parameters k_0/Δ and kv_F/Δ , see the Appendix A for more explicit details. Then one finds that the phonon dispersion law can be expressed as

$$E_k = c_s k (1 + \gamma k^2) + \mathcal{O}(k^5) , \quad (7)$$

where

$$c_s = \frac{v_F}{\sqrt{3}} , \quad \gamma = -\frac{v_F^2}{45\Delta^2} , \quad (8)$$

and c_s is the speed of sound of the system.

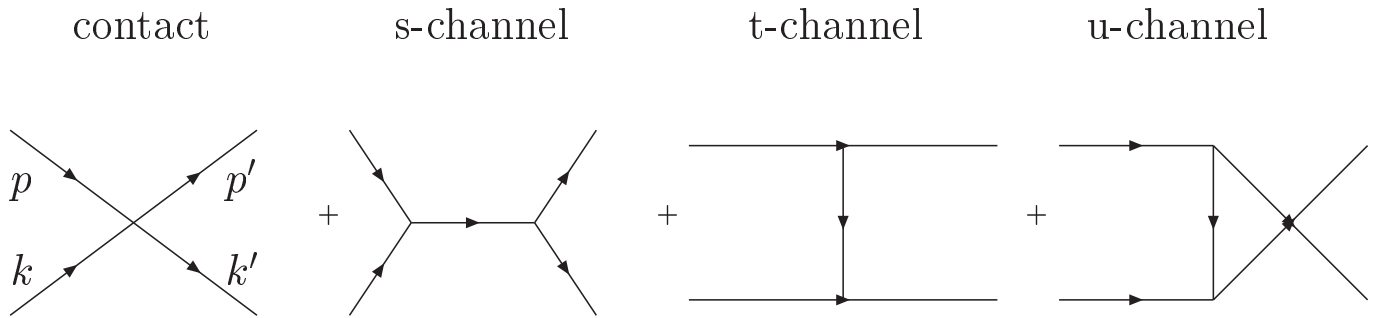


FIG. 2: Feynman diagrams contributing to $2 \leftrightarrow 2$ phonon scattering processes.

In most part of the core of the neutron stars, neutrons are supposed to condense in a 3P_2 channel. A similar computation to the one performed here could be used to determine the phonon dispersion law. Then one should keep in the neutron Lagrangian the interaction responsible for the formation of the gap, using the neutron propagator in the p-wave phase. Such a computation has been carried out in Ref. [22], but only the linear part of the phonon dispersion law has been obtained. In this paper we will not compute the correction to the linear term but simply assume that the correction is expressed as in the s-wave phase, using for Δ the angle-averaged value of the p-wave gap.

B. Relevant phonon collisions for the thermal conductivity

In this Subsection we briefly recall the relevant phonon collisions which are needed for the computation of the thermal conductivity. As the phonon dispersion law curves downward beyond linear order collisional processes of one phonon decaying into two or more phonons are kinematically forbidden (see the Appendix A of Ref. [2] for a brief discussion on this point). Thus, the first leading processes relevant for the thermal conductivity are binary collisions of phonons, see Fig. 2. Expressions for the different scattering amplitudes relevant for the binary collisions were explicitly given in Ref. [1]. We will not write them here again, but comment on the change in the amplitudes we will be using here with respect to those used in the computation of the shear viscosity of Ref. [1].

It turns out that if one naively uses the scattering amplitudes obtained from the LO effective field theory, then collinear singularities arise in the evaluation of the different transport coefficients [1, 2]. One then has to regulate the phonon propagators appearing in the different scattering amplitudes. For the computation of the shear viscosity, we regulated the phonon propagator with the one-loop thermal phonon damping [1, 23]. But we also checked that a different possible regularization, using the non-linear phonon dispersion law, was possible leading to the same results. This is due to the fact that the shear viscosity is dominated by the large angle collisions, as the viscosity describes transport in the direction orthogonal to the flow. In the computation of the thermal conductivity we see that the final result depends on large angle as well as on collinear (or small angle) collisions. In the collinear regime, the phonon propagators in the scattering amplitudes becomes almost on-shell. It has been checked that the imaginary part of the phonon self-energy vanishes when evaluated on-shell for $\gamma < 0$ [18]. Because in superfluid neutron matter we have found $\gamma < 0$, we are forced to regulate the phonon propagators appearing in the evaluation of the scattering amplitudes by taking the phonon dispersion beyond linear order. That is, we will be using the same scattering amplitudes that were used in Ref. [1], but with the phonon propagator expressed as

$$G(k_0, \mathbf{k}) = \frac{1}{k_0^2 - E_k^2}, \quad (9)$$

with E_k given by Eq. (7).

III. THERMAL CONDUCTIVITY

In this Section we present the results for the phonon contribution to the thermal conductivity in superfluid β -stable neutron matter. We start by summarizing the variational formalism we use to solve the Boltzmann equation [10]. We then present a scaling argument to obtain the temperature and gap dependences of the thermal conductivity,

which are determined by the leading contribution to the thermal conductivity coming from large and/or small angle phonon-phonon collisions. We finish this Section by showing the results of the numerical computations of the thermal conductivity.

A. Variational computation of the thermal conductivity

In hydrodynamics the thermal conductivity κ is the coefficient that relates the heat flux with a temperature gradient as

$$\mathbf{q} = -\kappa \nabla T . \quad (10)$$

In kinetic theory the heat flow can be expressed as

$$\mathbf{q} = \int \frac{d^3 p}{(2\pi)^3} \mathbf{v}_p E_p \delta f_p , \quad (11)$$

where $\mathbf{v}_p = \partial E_p / \partial \mathbf{p}$ is the particle velocity, and $\delta f_p = f_p - f_p^0$ is the deviation of the distribution function from the local thermal equilibrium given by

$$f_p^0 = \frac{1}{e^{p_\mu u^\mu / T} - 1} , \quad (12)$$

and u_μ is the local fluid velocity.

For the computation of the thermal conductivity one typically expresses the deviation from equilibrium as

$$\delta f_p = -\frac{f_p^0(1 + f_p^0)}{T^3} g(p) \mathbf{p} \cdot \nabla T , \quad (13)$$

and $g(p)$ is a dimensionless function. Thus, the thermal conductivity can be expressed as

$$\kappa = \frac{1}{3T^3} \int \frac{d^3 p}{(2\pi)^3} f_p^0(1 + f_p^0) g(p) v_p E_p p . \quad (14)$$

A solution to the Boltzmann equation should be respectful with the constraints of both energy and momentum conservation, thus

$$\int d\Gamma E_p \delta f_p = \int d\Gamma \mathbf{p} \delta f_p = 0 , \quad (15)$$

where $d\Gamma = d^3 p / (2\pi)^3$. Following these constraints, the authors of Ref. [10] realized that with a simple linear dispersion law the phonon contribution to the thermal conductivity would vanish. This result is consistent with the same conclusion first found out for superfluid helium [15]. Thus, in this work we will compute the thermal conductivity assuming a phonon dispersion law given by Eq. (7). The phonon velocity is therefore $v_p = c_s (1 + 3\gamma p)$.

To evaluate the thermal conductivity coefficient one considers the Boltzmann equation, with a collision term that takes into account the binary collisions discussed in Sec. II B, and linearizes it in the deviations δf_p . After some straightforward algebra, it is possible to check that this transport coefficient can alternatively be expressed as

$$\kappa = \frac{1}{12T^4} \int_{pkp'k'} (2\pi)^4 \delta(P + K - K' - P') |\mathcal{M}|^2 f_p^0 f_k^0 (1 + f_{k'}^0) (1 + f_{p'}^0) \Delta_g^2 , \quad (16)$$

where $P^\mu = (E_p, \mathbf{p})$, and

$$\Delta_g = g(p) \mathbf{p} + g(k) \mathbf{k} - g(k') \mathbf{k}' - g(p') \mathbf{p}' , \quad (17)$$

and we have defined the shorthand

$$\int_q \equiv \int \frac{d^3 q}{(2\pi)^3 2E_q} . \quad (18)$$

The expressions given in Eqs. (14) and (16) should be equivalent, provided $g(p)$ is a solution of the Boltzmann equation. If we use a variational approach, this equivalence allows us to find a closed expression for κ . We follow here

the same technique and form of the variational solution of Ref. [10]. Thus, we express the solution in terms of a basis of orthogonal polynomials

$$g(p) = \sum_{s=0}^{\infty} b_s B_s(p^2), \quad (19)$$

where $B_s(p^2)$ is a polynomial in p^2 of order s . The coefficient of the lowest power is set to 1, $B_0 = 1$. The polynomials $B_s(p^2)$ are orthogonal with each other,

$$\int d\Gamma f_p^0 (1 + f_p^0) p^2 B_s(p^2) B_t(p^2) \equiv A_s \delta_{st}. \quad (20)$$

With the help of $B_0 = 1$ all higher polynomials can be solved.

Further, demanding that the two expressions of the thermal conductivity be equivalent allows one to determine its value as

$$\kappa = \left(\frac{4a_1^2}{3T^2} \right) A_1^2 M_{11}^{-1}, \quad (21)$$

where M_{11}^{-1} is the (1,1) element of the inverse of the infinite matrix

$$M_{st} = \int d\Gamma_{p,k,k',p'} \mathbf{Q}_s \cdot \mathbf{Q}_t, \quad \mathbf{Q}_s = B_s(p^2) \mathbf{p} + B_s(k^2) \mathbf{k} - B_s(k'^2) \mathbf{k}' - B_s(p'^2) \mathbf{p}', \quad (22)$$

with

$$\int d\Gamma_{p,k,k',p'} \equiv \int_{pkp'k'} (2\pi)^4 \delta(P + K - K' - P') |\mathcal{M}|^2 f_p^0 f_k^0 (1 + f_{k'}^0) (1 + f_{p'}^0), \quad (23)$$

and for the superfluid phonons with a non-linear dispersion law one has

$$a_1 = \frac{4c_s^4}{15\Delta^2}, \quad A_1 = \frac{256\pi^6}{245c_s^9} T^9. \quad (24)$$

While the matrix M is a matrix of infinite dimensions, the variational treatment is performed by restricting the dimension of the matrix to be of order $N \times N$. Then one can prove that

$$\kappa \geq \left(\frac{4a_1^2}{3T^2} \right) A_1^2 M_{11}^{-1}, \quad (25)$$

where now M is a $N \times N$ dimensional matrix. The number N is treated then as a variational parameter in our numerical study.

B. Temperature dependence of the thermal conductivity

Once the scattering matrix \mathcal{M} is known, it is possible to determine the scaling behavior of κ with the temperature and the gap, depending whether small or large angle collisions give the leading contribution (for a similar discussion for the bulk viscosity please see Sec. 4 of Ref. [2]). From Eq. (25) it is possible to infer that $\kappa \propto \frac{T^{16}}{\Delta^4} M_{11}^{-1}$. Assuming $N = 1$ (for higher values of N a similar reasoning gives the same results) the scaling behavior of M_{11} is obtained by defining dimensionless variables $x_p = \frac{c_s p}{T}$, etc. Using these dimensionless variables, also to express the scattering matrix, the T dependence of the integrals appearing in Eq. (22) factorizes. In particular, for large angle collisions it is possible to check that $|\mathcal{M}|^2 \propto T^8$, and thus

$$\kappa \propto \frac{1}{T^2} \frac{1}{\Delta^4} \quad \text{for large angle collisions.} \quad (26)$$

For small angle collisions the scaling behavior of the scattering matrix is different, as then the phonon propagator in the amplitudes of binary collisions mediated by phonon exchange has to be considered in the collinear limit, and it is dominated by the non-linear piece in the phonon dispersion law. In this collinear limit the phonon propagator then

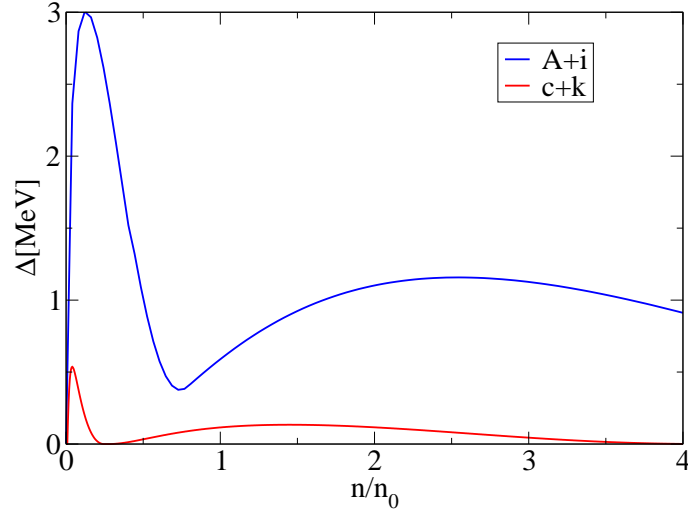


FIG. 3: (Color online) The 1S_0 and averaged 3P_2 neutron gaps as a function of the nucleon particle density in units of saturation density, $n_0 = 0.16 \text{ fm}^{-3}$. We use two very different gap models as a function of the density to illustrate the gap dependence of our results. In the first scheme we consider the $^1S_0(A) + ^3P_2(i)$ model, where the 1S_0 neutron gap is calculated in the BCS approach using different bare nucleon–nucleon interactions that converge towards a maximum neutron gap of about 3 MeV at $p_F \approx 0.85 \text{ fm}^{-1}$ (parametrization A of Table I in Ref. [14]) while for 3P_2 we have taken the parametrization i (strong neutron superfluidity in the core). The second model is the $^1S_0(c) + ^3P_2(k)$, where for 1S_0 channel the calculation of the gap goes beyond the BCS theory and in the 3P_2 channel we have taken into account a parametrization assuming weak neutron superfluidity in the core with maximum values for the gap of the order of 0.1 MeV [14].

scales as $1/(\gamma T^4) \propto 1/(T^4/\Delta^2)$ instead of $1/T^2$ as in the large angle case (see Ref. [2] for a more detailed explanation). The change of scaling of the photon propagator then leads to

$$\kappa \propto T^2 \frac{1}{\Delta^8} \quad \text{for small angle collisions.} \quad (27)$$

It is still possible to consider the scaling behavior for $|\mathcal{M}|^2$, where \mathcal{M} and \mathcal{M}^* corresponds to a small and large angle collision, respectively, or vice versa. Then the scaling arguments tells us that

$$\kappa \propto \frac{1}{\Delta^6} \quad \text{for combined large – small angle collisions.} \quad (28)$$

We note here that in the color-flavor-locked superfluid an explicit computation of the phonon contribution to the thermal conductivity gives the T -independent behavior mentioned above [10], while for the superfluid Fermi gas in the unitarity limit the phonon contribution to the thermal conductivity is dominated by small angle collisions [11], although in that case $\gamma > 0$. In our case, a numerical study is mandatory in order to obtain both the T scaling behavior and numerical value of the thermal conductivity for superfluid neutron matter. We present the results of such an analysis in the following Subsection.

C. Numerical values for the phonon contribution to the thermal conductivity in superfluid neutron matter

In order to calculate the thermal conductivity due to phonons in superfluid β -stable neutron matter two essential ingredients are needed, the EoS of β -equilibrated neutron star matter and the value of the gap. For the first, as already mentioned in Sec. I, we use the APR EoS for β -stable neutron matter. The other crucial ingredient is the value of the gap of superfluid neutron matter for densities inside the core of a neutron star.

The gap strongly depends on the nucleon–nucleon interaction model as well as on the many-body method used to compute it. Thus, there is not a consensus on its value and density dependence. In this work we use the 1S_0 and averaged 3P_2 neutron gaps shown in Fig. 3. We consider two very different gap models as a function of the density in order to illustrate the gap dependence of our results.

Our first model takes into account the maximum values that the gap for superfluid neutron matter could have in the relevant 1S_0 and 3P_2 channels. The gap model, named hereafter $^1S_0(A) + ^3P_2(i)$, consists of the 1S_0 neutron

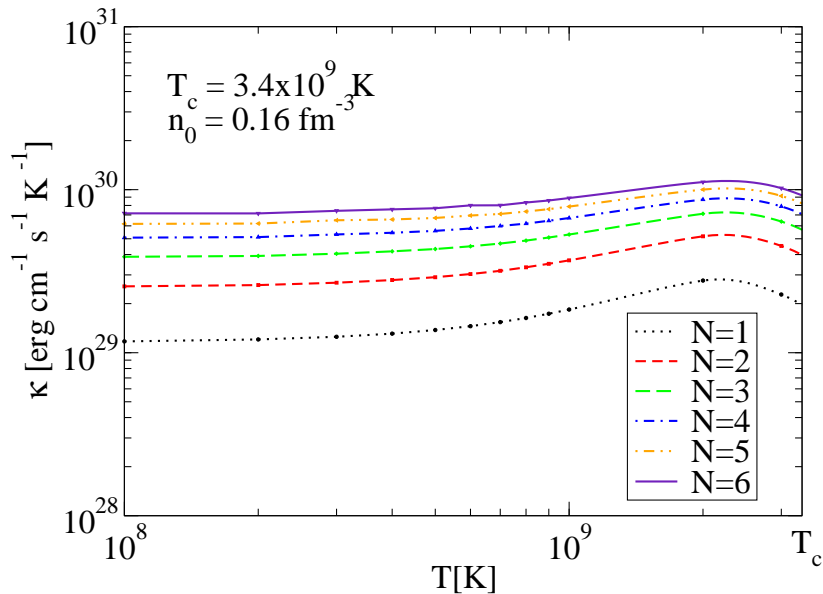


FIG. 4: (Color online) Variational calculation for the thermal conductivity due to superfluid phonons up to order $N = 6$ for nuclear matter saturation density, n_0 , as a function of temperature. The end temperature is the critical temperature which amounts for $T_c = 0.57\Delta(n_0) = 3.4 \times 10^9$ K. The calculation is performed for the $^1S_0(A)+^3P_2(i)$ model for the gap.

gap that results from the BCS approach using different bare nucleon–nucleon interactions that converge towards a maximum gap of about 3 MeV at $p_F \approx 0.85\text{fm}^{-1}$ (parametrization A of Table I in Ref. [14]). The anisotropic 3P_2 neutron gap is more challenging and not fully understood as one must extend BCS theory and calculate several coupled equations while including relativistic effects, since the gap extends for densities inside the core. We have taken the parametrization i (strong neutron superfluidity in the core) of Table I in Ref. [14] for the 3P_2 neutron angular averaged value, which presents a maximum value for the gap of approximately 1 MeV.

The second model considered is the $^1S_0(c)+^3P_2(k)$ scheme [14]. In this case, for 1S_0 channel, the calculation of the gap goes beyond the BCS theory by adding corrections to the bare nucleon-nucleon potential such as introducing an effective neutron mass modified by the presence of the small proton fraction. For the 3P_2 channel we consider a parametrization assuming weak neutron superfluidity in the core with maximum values for the gap of the order of 0.1 MeV. The reason for choosing this last model lies on the fact that values for the gap of 0.1 MeV were claimed to be inferred from the fast cooling of CasA [24], although later analysis have questioned these results [25, 26].

With all these ingredients, we can proceed to calculate the thermal conductivity due to superfluid phonons using the variational method described in Sec. III A. The evaluation of the thermally weighted scattering matrix for phonons in Eq. (22) is performed numerically using the *Vegas* Monte Carlo algorithm [27]. In Fig. 4 we show the results up to order $N = 6$ for nuclear matter saturation density, $n_0 = 0.16\text{fm}^{-3}$, as a function of temperature. The final value of the number N is determined by imposing that the deviation with respect to the previous order should be $\lesssim 10\%$. The end temperature is the critical temperature which amounts for $T_c = 0.57\Delta(n_0) = 3.4 \times 10^9$ K, with $\Delta(n)$ given in Fig. 3 for the $^1S_0(A)+^3P_2(i)$ model for the gap. Similar results for thermal conductivity are obtained using the $^1S_0(c)+^3P_2(k)$ gap function.

Close to T_c we would expect that higher order corrections in the energy and momentum expansion should be taken into account both in the phonon dispersion law and self-interactions. For $T \lesssim 10^9$ K, below T_c , the thermal conductivity is almost independent of the temperature, with subleading temperature corrections of order T and T^2 , respectively. In the particular case of Fig. 4 and $N = 6$, we find that a fit to our numerical results is given by $\kappa \sim (7.02 \times 10^{29} + 9.28 \times 10^{19} T + 9.08 \times 10^{10} T^2) \text{ erg cm}^{-1} \text{ s}^{-1} \text{ K}^{-1}$, with T given in Kelvin. In fact, the comparison between our numerical results for the thermal conductivity for both gaps and the previous scaling arguments for the thermal conductivity with the temperature show that the dominant processes to the phonon contribution to the thermal conductivity at low T are those corresponding to the combined small and large angle collisions, as discussed in Eq. (28), leading to an almost independent behaviour of the thermal conductivity with temperature. Then, the thermal conductivity scales as $\kappa \propto 1/\Delta^6$ at low T , the factor of proportionality depending on the density. Note that this is the same behavior found for the color-flavor-locked superfluid [10], although in that case the factor of proportionality could be expressed in terms of the quark chemical potential.

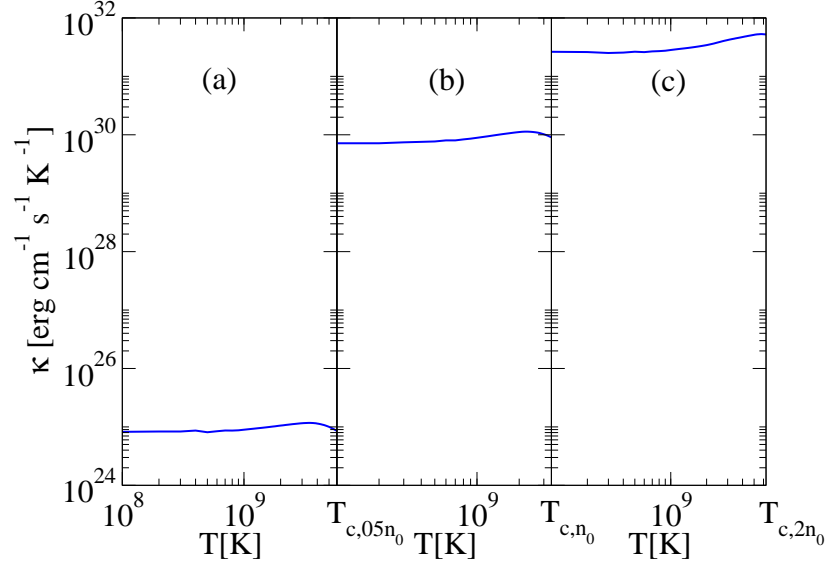


FIG. 5: (Color online) Phonon contribution to the thermal conductivity using the $^1S_0(A)+^3P_2(i)$ model for the gap as a function of temperature for different particle densities in units of normal saturation density n_0 : (a) for $0.5n_0$ being $T_{c,0.5n_0} = 5.8 \times 10^9$ K, (b) for n_0 being $T_{c,n_0} = 3.4 \times 10^9$ K and (c) for $2n_0$ being $T_{c,2n_0} = 6.3 \times 10^9$ K. The end temperature for each density is the critical temperature, $T_c = 0.57\Delta$.

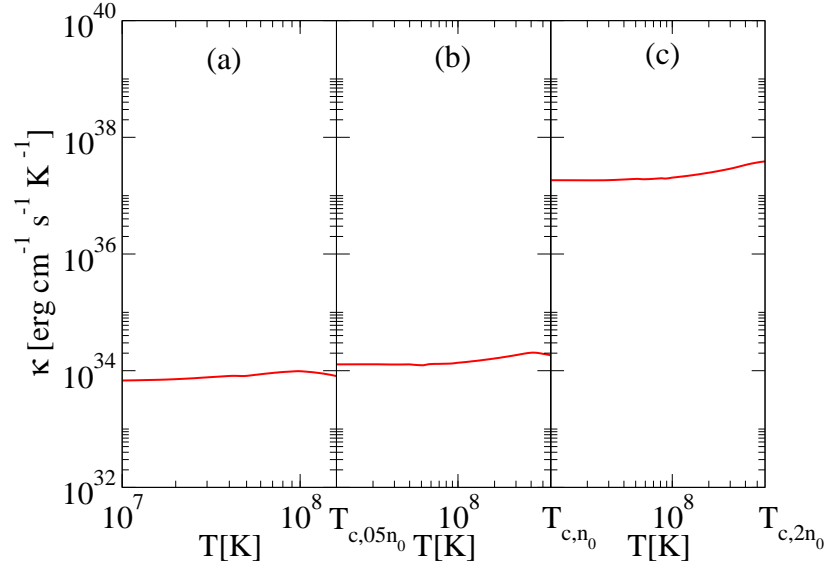


FIG. 6: (Color online) The same as in Fig. 5, but using the $^1S_0(c)+^3P_2(k)$ model for the gap. In this case, the different particle densities in units of normal saturation density n_0 and critical temperatures are: (a) $0.5 n_0$ with $T_{c,0.5n_0} = 1.6 \times 10^8$ K, (b) n_0 with $T_{c,n_0} = 5.8 \times 10^8$ K and (c) $2n_0$ with $T_{c,2n_0} = 5.8 \times 10^8$ K.

In Figs. 5 and 6 we display the contribution of phonon-phonon scattering to the thermal conductivity as a function of temperature for different particle densities in units of n_0 for the two models of the gap previously discussed, respectively. We start from densities of $0.5 n_0$, as the lower density one can reach in the core of the neutron star. The end temperature for each density is the critical temperature, $T_c = 0.57\Delta$, with the gap for different densities given in Fig. 3. The thermal conductivity grows with increasing density, with a non-linear dependence according to the evolution with density of the gap.

From our results, one can also extract the thermal conductivity mean free path of the phonons, which is not the same as the mean free path associated to shear viscosity (see Ref. [1] for a comparison). For the thermal conductivity

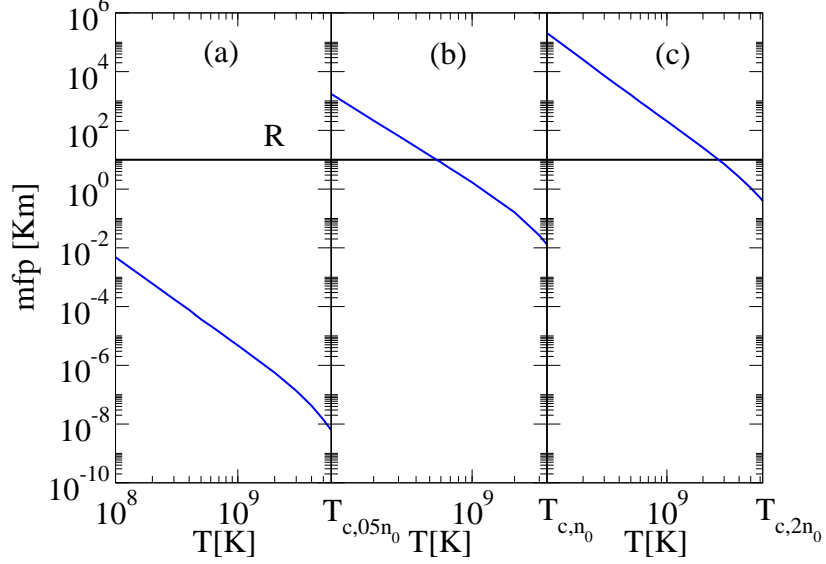


FIG. 7: (Color online) Mean-free path due to the phonon thermal conductivity using the $^1S_0(A)+^3P_2(i)$ model for the gap as a function of temperature for three different densities in units of n_0 : (a) for $0.5n_0$ being $T_{c,0.5n_0} = 5.8 \times 10^9$ K, (b) for n_0 being $T_{c,n_0} = 3.4 \times 10^9$ K and (c) for $2n_0$ being $T_{c,2n_0} = 6.3 \times 10^9$ K. The end temperature for each density is the critical temperature, $T_c = 0.57\Delta$. The mean-free path has to be compared with the radius of the star, which we take $R = 10$ Km.

the mean free path (mfp) is defined as

$$l = \frac{\kappa}{\frac{1}{3}c_v c_s}, \quad (29)$$

with the heat capacity for phonons given by [15]

$$c_v = \frac{2\pi^2}{15c_s^3} \left(T^3 + \frac{25\gamma}{7} \frac{(2\pi)^2}{c_s^2} T^5 \right). \quad (30)$$

We show in Fig. 7 the mfp of phonons in β -stable neutron star matter for $0.5n_0$, n_0 and $2n_0$ as a function of temperature using the $^1S_0(A)+^3P_2(i)$ model for the gap. We also indicate an estimate of the limit of 10 Km for the radius of the star. We observe that for $n = 0.5n_0$ the superfluid phonon mfp stays below the radius of the star, also for $n = n_0$ and $T \gtrsim 6 \times 10^8$ K as well as for $2n_0$ and $T \gtrsim 3 \times 10^9$ K. As for the $^1S_0(c)+^3P_2(k)$ gap function, we obtain that the superfluid phonon mfp for the temperatures and densities studied are too large, as the thermal conductivity is several orders of magnitude higher than the previous case. Thus, we only show numerical results for our first gap. We find that $l \propto 1/T^3$. The temperature dependence of the mfp results from the almost temperature independent behaviour of the thermal conductivity.

A relevant discussion is to know the temperature regime where our hydrodynamical results are valid. As mentioned previously, close to T_c we would expect that higher order corrections in the energy and momentum expansion should be taken into account both in the phonon dispersion law and self-interactions, but these have not been derived yet. Further, the density of superfluid phonons becomes very dilute at very low T , and then it might be difficult to maintain a hydrodynamical description of their behavior, if the phonons do not collide with other light particles, such as the electrons. Otherwise, the phonons would behave in the low T regime ballistically. In such a case the thermal conductivity due to phonons would be then dominated by the collisions of the phonons with the boundary. The behavior of the thermal conductivity in this regime can be qualitatively estimated as

$$\kappa = \frac{1}{3}c_v c_s R, \quad (31)$$

with R being the radius of the superfluid core of the star. In this case $\kappa \propto T^3$, and it would clearly drop at sufficiently low T . This ballistic behavior of the phonons of superfluid ^4He has been experimentally tested [28]. However, the process of heat transport in a ballistic regime follows a different law than Eq. (10) (see Refs. [29–31]). In such a case, a more detailed analysis of the Boltzmann equation obeyed by the phonons should be carried out. Prior to this, one

should analyze whether electron-superfluid phonon interactions could be more important than the phonon collisions with the boundary and whether these may give a relevant contribution to the thermal conductivity. This will be the subject of future studies.

IV. CONCLUSIONS

We have carried out a numerical computation of the phonon contribution to the thermal conductivity in superfluid β -stable neutron matter. We have obtained the phonon scattering rates using effective field theory techniques in terms of the EoS of the system. For that purpose, we have used a causal parametrization of a widely used benchmark, the APR EoS. Moreover, since the thermal conductivity vanishes using a linear phonon dispersion law, we have calculated the first correction beyond linear order, which depends on the gap of neutron matter. Two different models for the gap in the strong and weak superfluidity regimes have been considered. With all these ingredients, we have solved the Boltzmann equation using a variational approach and have checked that, as it occurs for the color-flavor locked superfluid, it can be well approximated for $T \ll T_c$ by an almost T independent result, where $\kappa \propto 1/\Delta^6$, the factor of proportionality depending on the density and EoS of the superfluid. Our analysis indicates that this behavior arises by the combined effect of large and small angle phonon binary collisions.

Our results should be compared to the thermal conductivity of electrons and muons mediated by electromagnetic interactions in neutron star cores. In Ref. [8] the calculations of the electron contribution to the thermal conductivity has been revised, considering the effect of the Landau damping of electromagnetic interactions owing to the exchange of transverse plasmons as well as the presence of proton superconductivity that had been previously neglected [6, 7]. According to Fig. 5 of Ref. [8], for a mass density of $\rho = 4 \times 10^{14} \text{ g cm}^{-3}$, the thermal conductivity of electrons and muons, $\kappa_{e\mu}$, has values of $\kappa_{e\mu} \sim 10^{23-24} \text{ erg cm}^{-1} \text{ s}^{-1} \text{ K}^{-1}$ for $T \sim 10^8 \text{ K}$ depending on the strength of the proton superconductivity, while $\kappa_{e\mu}$ is reduced by few orders of magnitude at $T \sim 10^9 \text{ K}$ following the asymptotic law, $\kappa_{e\mu} \propto T^{-1}$. For lower mass densities and in the limit of strong superfluidity, we could easily estimate the thermal conductivity using Eq. (93) of Ref. [8]. In this case we find that for a mass density of $\rho \sim 1.4 \times 10^{14} \text{ g cm}^{-3}$ (that corresponds to the lowest density inside the core of $n \sim 0.5n_0$) the thermal conductivity of electrons and muons is reduced by a factor three at most. Thus, we find that thermal conductivity in the neutron star core is dominated by phonon-phonon collisions when phonons are in a pure hydrodynamical regime, since $10^{25} \lesssim \kappa_{ph} \lesssim 10^{32} \text{ erg cm}^{-1} \text{ s}^{-1} \text{ K}^{-1}$ from $0.5n_0$ to $2n_0$ for temperatures up to T_c .

Our results also indicate that if the contribution of electrons-muons and phonons to κ get comparable, electron-phonon collisions could play an important role in the determination of κ . It is possible to study those interactions using also the effective field theory techniques [3]. If the electron-phonon collisions are relevant, one would then be forced to perform a study of the thermal conductivity in an electron-phonon coupled system. Simple estimates have been done in Ref. [32]. This topic should be carefully analyzed in future studies.

Appendix A: Polarization tensor

We show here some intermediate steps and details for the computation phonon dispersion law. The computation of $\Pi_{\mu\nu}$ is similar to the computation of the one-loop polarization tensor associated to a $U(1)$ gauge field (see also Ref. [21]). Starting from Eq. (5), we perform the trace over the indices in the Nambu-Gorkov space. Then one can express the different (symmetric) components of the polarization tensor as follows

$$\Pi_{00}(q_0, \mathbf{q}) = -2i \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{p_0 k_0 + \xi_p \xi_k - \Delta^2}{(p_0^2 - \xi_p^2 - \Delta^2)(k_0^2 - \xi_k^2 - \Delta^2)} \right\}, \quad (\text{A1})$$

$$\Pi_{ij}(q_0, \mathbf{q}) = -2i \int \frac{dp_0}{(2\pi)} \frac{d^3 p}{(2\pi)^3} \left\{ v_i v_j \frac{p_0 k_0 + \xi_p \xi_k + \Delta^2}{(p_0^2 - \xi_p^2 - \Delta^2)(k_0^2 - \xi_k^2 - \Delta^2)} \right\}, \quad (\text{A2})$$

$$\Pi_{0i}(q_0, \mathbf{q}) = -2i \int \frac{dp_0}{(2\pi)} \frac{d^3 p}{(2\pi)^3} v_i \left\{ \frac{p_0 \xi_k + k_0 \xi_p}{(p_0^2 - \xi_p^2 - \Delta^2)(k_0^2 - \xi_k^2 - \Delta^2)} \right\}, \quad (\text{A3})$$

where $k_0 = p_0 - q_0$ and $\xi_k = \xi_{(p-q)}$, and we have considered that the velocities are $\mathbf{v}_k = \mathbf{v}_{(p-q)} = \mathbf{v}$.

The p_0 integral can be performed by contour integration, assuming retarded boundary conditions $p_0 \rightarrow p_0 + i0^+ \text{sign}(p_0)$. The remaining angular integrals can be easily performed if we only compute $q^\mu q^\nu \Pi_{\mu\nu}(q_0, q)$. The final integral in p can then easily performed if we assume that $\mu/\Delta \rightarrow \infty$, which is a good approximation given the values

of the neutron gaps and chemical potential. Then one finds

$$\Pi_{00}(q_0, q) = \frac{2p_F m^*}{\pi^2} \left(1 + \frac{q_0^2}{6\Delta^2} + \mathcal{O}\left(\frac{q_0^4}{\Delta^4}, \frac{q^4 v_F^4}{\Delta^4}\right) \right), \quad (\text{A4})$$

$$q^i q^j \Pi_{ij}(q_0, q) = \frac{p_F m^* q^2 v_F^2}{30\pi^2} \left(\frac{q^2 v_F^2}{\Delta^2} + \mathcal{O}\left(\frac{q_0^4}{\Delta^4}, \frac{q^4 v_F^4}{\Delta^4}\right) \right), \quad (\text{A5})$$

$$q^i \Pi_{0i}(q_0, q) = \frac{p_F m^* q v_F}{2\pi^2} \left(\frac{1}{9} \frac{q_0 q v_F}{\Delta^2} + \mathcal{O}\left(\frac{q_0^4}{\Delta^4}, \frac{q^4 v_F^4}{\Delta^4}\right) \right), \quad (\text{A6})$$

which are the values needed to compute the phonon dispersion law at the order of accuracy required in this manuscript.

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